

Space–time percolation

Geoffrey Grimmett*

February 1, 2008

Abstract

The contact model for the spread of disease may be viewed as a directed percolation model on $\mathbb{Z} \times \mathbb{R}$ in which the continuum axis is oriented in the direction of increasing time. Techniques from percolation have enabled a fairly complete analysis of the contact model at and near its critical point. The corresponding process when the time-axis is unoriented is an undirected percolation model to which now standard techniques may be applied. One may construct in similar vein a random-cluster model on $\mathbb{Z} \times \mathbb{R}$, with associated continuum Ising and Potts models. These models are of independent interest, in addition to providing a path-integral representation of the quantum Ising model with transverse field. This representation may be used to obtain a bound on the entanglement of a finite set of spins in the quantum Ising model on \mathbb{Z} , where this entanglement is measured via the entropy of the reduced density matrix. The mean-field version of the quantum Ising model gives rise to a random-cluster model on $K_n \times \mathbb{R}$, thereby extending the Erdős–Rényi random graph on the complete graph K_n .

1 Introduction

Brazil is justly famous for its beach life and its probability community. In harnessing the first to support the second, a summer school of intellectual distinction and international visibility in probability theory has been created. The high scientific stature of the organizers and of the wider Brazilian community has ensured the attendance of a host of wonderful lecturers during ten years of the Brazilian School of Probability, and the School has attracted an international audience including many young Brazilians who continue to

*Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK

leave their marks within this crossroads subject of mathematics. The warmth and vitality of Brazilian culture have been attractive features of these summer schools, and invitations to participate are greatly valued. This short review concerns two topics of recurring interest at the School, namely percolation and the Ising model (in both its classical and quantum forms), subject to the difference that one axis of the underlying space is allowed to vary continuously.

The percolation process is arguably the most fundamental of models for a disordered medium. Its theory is now well established, and several mathematics books have been written on and near the topic, see [14, 16, 24, 44]. Percolation is at the source of one of the most exciting areas of contemporary probability theory, namely the theory of Schramm–Löwner evolutions (SLE). This theory threatens to explain the relationship between probabilistic models and conformal field theory, and is expected to lead ultimately to rigorous explanations of scaling theory for a host of two dimensional models including percolation, self-avoiding walks, and the Ising/Potts and random-cluster models. See [34, 43, 45, 46].

Percolation theory has contributed via the random-cluster model to the study of Ising/Potts models on a given graph G , see [25]. The methods developed for percolation have led also to solutions of several of the basic questions about the contact model on $G \times \mathbb{R}$, see [1, 7, 8, 36]. It was shown in [2] that the quantum Ising model with transverse field on G may be reformulated in terms of a random-cluster model on $G \times \mathbb{R}$, and it has been shown recently in [27] that random-cluster arguments may be used to study entanglement in the quantum Ising model.

In this short account of percolative processes on $G \times \mathbb{R}$ for a lattice G , we shall recall in Sections 2–3 the problems of percolation on $G \times \mathbb{R}$, and of the contact model on G . This is followed in Section 4 by a description of the continuum random-cluster model on $G \times \mathbb{R}$, and its application to continuum Ising/Potts models. In Section 5 we present a summary of the use of random-cluster techniques to study entanglement in the quantum Ising model on \mathbb{Z} . An account is included of a recent result of [27] stating that the entanglement entropy of a line of L spins has order not exceeding $\log L$ in the strong-field regime. The proof relies on a property of random-cluster measures termed ‘ratio weak-mixing’, studied earlier in [3, 4] for the random-cluster model on a lattice. The corresponding mean-field model is considered in Section 6.

2 Continuum percolation

Let $G = (V, E)$ be a finite or countably infinite graph which, for simplicity, we take to be connected with neither loops nor multiple edges. We shall usually

take G to be a subgraph of the hypercubic lattice \mathbb{Z}^d for some $d \geq 1$. The models of this paper inhabit the space $G \times \mathbb{R}$, which we refer to as space-time, and we think of $G \times \mathbb{R}$ as being obtained by attaching a ‘time-line’ $(-\infty, \infty)$ to each vertex $x \in V$.

Let $\lambda, \delta \in (0, \infty)$. The continuum percolation model on $G \times \mathbb{R}$ is constructed via processes of ‘cuts’ and ‘bridges’ as follows. For each $x \in V$, we select a Poisson process D_x of points in $\{x\} \times \mathbb{R}$ with intensity δ ; the processes $\{D_x : x \in V\}$ are independent, and the points in the D_x are termed ‘cuts’. For each $e = \langle x, y \rangle \in E$, we select a Poisson process B_e of points in $\{e\} \times \mathbb{R}$ with intensity λ ; the processes $\{B_e : e \in E\}$ are independent of each other and of the D_x . Let $\mathbb{P}_{\lambda, \delta}$ denote the probability measure associated with the family of such Poisson processes indexed by $V \cup E$.

For each $e = \langle x, y \rangle \in E$ and $(e, t) \in B_e$, we think of (e, t) as an edge joining the endpoints (x, t) and (y, t) , and we refer to this edge as a ‘bridge’. For $(x, s), (y, t) \in V \times \mathbb{R}$, we write $(x, s) \leftrightarrow (y, t)$ if there exists a path π with endpoints $(x, s), (y, t)$ such that: π comprises cut-free sub-intervals of $G \times \mathbb{R}$ together with bridges. For $\Lambda, \Delta \subseteq V \times \mathbb{R}$, we write $\Lambda \leftrightarrow \Delta$ if there exist $a \in \Lambda$ and $b \in \Delta$ such that $a \leftrightarrow b$.

For $(x, s) \in V \times \mathbb{R}$, let $C_{x,s}$ be the set of all points (y, t) such that $(x, s) \leftrightarrow (y, t)$. The clusters $C_{x,s}$ have been studied in [8], where the case $G = \mathbb{Z}^d$ was considered in some detail. Let 0 denote the origin $(0, 0) \in \mathbb{Z}^d \times \mathbb{R}$, and let $C = C_0$ denote the cluster at the origin. Noting that C is a union of line-segments, we write $|C|$ for the Lebesgue measure of C . The *radius* $\text{rad}(C)$ of C is given by

$$\text{rad}(C) = \sup\{\|x\| + |t| : (x, t) \in C\},$$

where

$$\|x\| = \sup_i |x_i|, \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d,$$

is the supremum norm on \mathbb{Z}^d .

The critical point of the process is defined by

$$\lambda_c(\delta) = \sup\{\lambda : \theta(\lambda, \delta) = 0\},$$

where

$$\theta(\lambda, \delta) = \mathbb{P}_{\lambda, \delta}(|C| = \infty).$$

It is immediate by time-scaling that $\theta(\lambda, \delta) = \theta(\lambda/\delta, 1)$, and we shall use the abbreviations $\lambda_c = \lambda_c(1)$ and $\theta(\lambda) = \theta(\lambda, 1)$.

The following exponential-decay theorem will be useful for the study of the quantum Ising model in Section 5.

Theorem 2.1. [8] *Let $G = \mathbb{Z}^d$ where $d \geq 1$, and consider continuum percolation on $G \times \mathbb{R}$.*

- (i) *We have that $\theta(\lambda_c) = 0$.*
- (ii) *Let $\lambda, \delta \in (0, \infty)$. There exist β, γ satisfying $\beta, \gamma > 0$ for $\lambda/\delta < \lambda_c$ such that:*

$$\mathbb{P}_{\lambda, \delta}(|C| \geq k) \leq e^{-\gamma k}, \quad k > 0, \quad (2.2)$$

$$\mathbb{P}_{\lambda, \delta}(\text{rad}(C) \geq k) \leq e^{-\beta k}, \quad k > 0. \quad (2.3)$$

- (iii) *When $d = 1$, $\lambda_c = 1$.*

The situation is rather different when the environment is chosen at random. With $G = (V, E)$ as above, suppose that the Poisson process of cuts at a vertex $x \in V$ has some intensity δ_x , and that of bridges parallel to the edge $e = \langle x, y \rangle \in E$ has some intensity λ_e . Suppose further that the δ_x , $x \in V$, are independent, identically distributed random variables, and the λ_e , $e \in E$ also. Write Δ and Λ for independent random variables having the respective distributions, and P for the probability measure governing the environment. [As before, $\mathbb{P}_{\lambda, \delta}$ denotes the measure associated with the percolation model in the given environment.]

If there exist $\lambda', \delta' \in (0, \infty)$ such that $\lambda'/\delta' < \lambda_c$ and $P(\Lambda \leq \lambda') = P(\Delta \geq \delta') = 1$, then the process is almost surely dominated by a subcritical percolation process, whence there is (almost sure) exponential decay in the sense of Theorem 2.1(ii). This may fail in an interesting way if there is no such almost-sure domination, in that one may prove exponential decay in the space-direction but only a weaker decay in the time-direction.

For any probability measure μ and function f , we write $\mu(f)$ for the expectation of f under μ . For $(x, s), (y, t) \in \mathbb{Z}^d \times \mathbb{R}$ and $q \geq 1$, we define

$$d_q(x, s; y, t) = \max\{\|x - y\|, [\log(1 + |s - t|)]^q\}.$$

Theorem 2.4. [32, 33] *Let $G = \mathbb{Z}^d$ where $d \geq 1$. Suppose that*

$$\Gamma = \max\{P([\log(1 + \Lambda)]^\beta), P([\log(1 + \Delta^{-1})]^\beta)\} < \infty,$$

for some $\beta > 2d^2(1 + \sqrt{1 + d^{-1}} + (2d)^{-1})$. There exists $Q = Q(d, \beta) > 1$ such that the following holds. For $q \in [1, Q)$ and $m > 0$, there exists $\epsilon = \epsilon(d, \beta, \Gamma, m, q) > 0$ and $\eta = \eta(d, \beta, q) > 0$ such that: if

$$E([\log(1 + (\Lambda/\Delta))]^\beta) < \epsilon,$$

there exist identically distributed random variables $D_x \in L^\eta(P)$, $x \in \mathbb{Z}^d$, such that

$$\mathbb{P}_{\lambda, \delta}((x, s) \leftrightarrow (y, t)) \leq \exp[-md_q(x, s; y, t)] \quad \text{if } d_q(x, s; y, t) \geq D_x,$$

for $(x, s), (y, t) \in \mathbb{Z}^d \times \mathbb{R}$.

The corresponding theorem of [32] contains no estimate for the tail of the D_x . The above moment property may be derived from the Borel–Cantelli argument used in the proof of [32], which proceeds by a so-called multiscale analysis, see [27], Section 8. Explicit values may be given for the constants Q and η , namely

$$Q = \frac{\beta(\alpha - d + \alpha d)}{\alpha d(\alpha + \beta + 1)},$$

where $\alpha = d + \sqrt{d^2 + d}$, and one may take η satisfying

$$(\eta + 1)\alpha < \frac{\beta}{\alpha} \left(\frac{\alpha - d + \alpha d}{q} - \alpha d \right) - d.$$

Complementary accounts of the *survival* of the process in a random environment may be found in [2, 5, 15, 39].

We mention two further types of ‘continuum’ percolation processes that arise in applications and have attracted the attention of probabilists. Let Π be a Poisson process of points in \mathbb{R}^d with intensity 1. Two points $x, y \in \Pi$ are joined by an edge, and said to be adjacent, if they satisfy a given condition of proximity. One now asks for conditions under which the resulting random graph possesses an unbounded component.

The following conditions of proximity have been studied in the literature.

1. *Lily-pond model*. Fix $r > 0$, and join x and y if and only if $|x - y| \leq r$, where $|\cdot|$ denotes Euclidean distance. There has been extensive study of this process, and of its generalization, the *random connection model*, in which x and y are joined with probability $g(|x - y|)$ for some given non-increasing function $g : (0, \infty) \rightarrow [0, 1]$. See [24, 38, 40].
2. *Voronoi percolation*. To each $x \in \Pi$ we associate the *tile*

$$T_x = \{z \in \mathbb{R}^d : |z - x| \leq |z - y| \text{ for all } y \in \Pi \setminus \{x\}\}.$$

Two tiles T_x, T_y are declared *adjacent* if their boundaries share a facet of a hyperplane of \mathbb{R}^d . We colour each tile *red* with probability ρ , different tiles receiving independent colours, and we ask for conditions under which there exists an infinite path of red tiles.

This model has a certain property of conformal invariance when $d = 2, 3$, see [6]. When $d = 2$, there is an obvious property of self-matching, leading to the conjecture that the critical point is given by $\rho_c = \frac{1}{2}$, and this has been proved recently in [13].

3 The contact model

Just as directed percolation on \mathbb{Z}^d arises by allowing only open paths that are ‘stiff’ in one direction, so the contact model on G is obtained from percolation on $G \times \mathbb{R}$ by requiring that open paths traverse time-lines in the direction of *increasing* time.

As before, we let D_x , $x \in V$, be Poisson processes with intensity δ , and we term points in the D_x ‘cuts’. We replace each $e = \langle x, y \rangle \in E$ by two oriented edges $[x, y]$, $[y, x]$, the first oriented from x to y , and the second from y to x . Write \vec{E} for the set of oriented edges thus obtained from E . For each $\vec{e} = [x, y] \in \vec{E}$, we let $B_{\vec{e}}$ be a Poisson process with intensity λ ; members of $B_{\vec{e}}$ are termed ‘directed bridges’ from x to y .

For $(x, s), (y, t) \in V \times \mathbb{R}$, we write $(x, s) \rightarrow (y, t)$ if there exists an oriented path π from (x, s) to (y, t) such that: π comprises cut-free sub-intervals of $V \times \mathbb{R}$ traversed in the direction of increasing time, together with directed bridges in the directions of their orientations. For $\Lambda, \Delta \subseteq V \times \mathbb{R}$, we write $\Lambda \rightarrow \Delta$ if there exist $a \in \Lambda$ and $b \in \Delta$ such that $a \rightarrow b$.

The directed cluster \vec{C} at the origin is the set

$$\vec{C} = \{(x, s) \in V \times \mathbb{R} : 0 \rightarrow (x, s)\},$$

of points reachable from the origin 0 along paths directed away from 0. The percolation probability is given by

$$\vec{\theta}(\lambda, \delta) = \mathbb{P}_{\lambda, \delta}(|\vec{C}| = \infty),$$

and the critical point by

$$\vec{\lambda}_c(\delta) = \sup\{\lambda : \vec{\theta}(\lambda, \delta) = 0\}.$$

As before, we write $\vec{\theta}(\lambda) = \vec{\theta}(\lambda, 1)$ and $\vec{\lambda}_c = \vec{\lambda}_c(1)$.

Parts (i) and (ii) of Theorem 2.1 are valid in this new setting, with C replaced by \vec{C} , etc, see [7]. The exact value of the critical point is unknown even when $d = 1$, although there are physical reasons to believe in this case that $\vec{\lambda}_c = 1.694\dots$, the critical value of the so-called reggeon spin model, see [23, 35]. The contact model in a random environment may be studied as in Theorem 2.4.

Further theory of the contact model may be found in [35, 36]. Sakai and Slade [42] have shown how to apply the lace expansion to the spread-out contact model on \mathbb{Z}^d for $d > 4$, and related results are valid for oriented percolation even when the connection function has unbounded domain, see [17, 18].

4 Random-cluster and Ising/Potts models

The percolation model on a graph $G = (V, E)$ may be generalized to obtain the random-cluster model on G , see [25]. Similarly, the continuum percolation model on $G \times \mathbb{R}$ may be extended to a continuum random-cluster model. Let W be a finite subset of V that induces a connected subgraph of G , and let E_W denote the set of edges joining vertices in W . Let $T \in (0, \infty)$, and let Λ be the ‘box’ $\Lambda = W \times [0, T]$. Let $\mathbb{P}_{\Lambda, \lambda, \delta}$ denote the probability measure associated with the Poisson processes D_x , $x \in W$, and B_e , $e = \langle x, y \rangle \in E_W$. As sample space we take the set Ω_Λ comprising all finite sets of cuts and bridges in Λ , and we may assume without loss of generality that no cut is the endpoint of any bridge. For $\omega \in \Omega_\Lambda$, we write $B(\omega)$ and $D(\omega)$ for the sets of bridges and cuts, respectively, of ω . The appropriate σ -field \mathcal{F}_Λ is that generated by the open sets in the associated Skorohod topology, see [8, 20].

For a given configuration $\omega \in \Omega_\Lambda$, let $k(\omega)$ be the number of its clusters under the connection relation \leftrightarrow . Let $q \in (0, \infty)$, and define the ‘continuum random-cluster’ probability measure $\mathbb{P}_{\Lambda, \lambda, \delta, q}$ by

$$d\mathbb{P}_{\Lambda, \lambda, \delta, q}(\omega) = \frac{1}{Z} q^{k(\omega)} d\mathbb{P}_{\Lambda, \lambda, \delta}(\omega), \quad \omega \in \Omega_\Lambda, \quad (4.1)$$

for an appropriate normalizing constant, or ‘partition function’, $Z = Z_\Lambda(\lambda, \delta, q)$. The quantity q is called the *cluster-weighting factor*. The continuum random-cluster model may be studied in very much the same way as the random-cluster model on a lattice, see [25].

The space Ω_Λ is a partially ordered space with order relation given by: $\omega_1 \leq \omega_2$ if $B(\omega_1) \subseteq B(\omega_2)$ and $D(\omega_1) \supseteq D(\omega_2)$. A random variable $X : \Omega_\Lambda \rightarrow \mathbb{R}$ is called *increasing* if $X(\omega) \leq X(\omega')$ whenever $\omega \leq \omega'$. An event $A \in \mathcal{F}_\Lambda$ is called *increasing* if its indicator function 1_A is increasing. Given two probability measures μ_1, μ_2 on a measurable pair $(\Omega_\Lambda, \mathcal{F}_\Lambda)$, we write $\mu_1 \leq_{\text{st}} \mu_2$ if $\mu_1(X) \leq \mu_2(X)$ for all bounded increasing continuous random variables $X : \Omega_\Lambda \rightarrow \mathbb{R}$.

The measures $\mathbb{P}_{\Lambda, \lambda, \delta, q}$ have certain properties of stochastic ordering as the parameters $\Lambda, \lambda, \delta, q$ vary. The basic theory will be assumed here, and the reader is referred to [9] for further details. In rough terms, the $\mathbb{P}_{\Lambda, \lambda, \delta, q}$ inherit

the properties of stochastic ordering and positive association enjoyed by their counterparts on discrete graphs. Of particular value later will be the stochastic inequality

$$\mathbb{P}_{\Lambda, \lambda, \delta, q} \leq_{\text{st}} \mathbb{P}_{\Lambda, \lambda, \delta} \quad \text{when } q \geq 1. \quad (4.2)$$

While it will not be important for what follows, we note that the thermodynamic limit may be taken in much the same manner as for the discrete random-cluster model, whenever $q \geq 1$. Suppose, for example, that W is a finite connected subgraph of the lattice $G = \mathbb{Z}^d$, and assign to the box $\Lambda = W \times [0, T]$ a suitable boundary condition. As in [25], if the boundary condition τ is chosen in such a way that the measures $\mathbb{P}_{\Lambda, \lambda, \delta, q}^\tau$ are monotonic as $W \uparrow \mathbb{Z}^d$, then the weak limit $\mathbb{P}_{\lambda, \delta, q, T}^\tau = \lim_{W \uparrow \mathbb{Z}^d} \mathbb{P}_{\Lambda, \lambda, \delta, q}^\tau$ exists. One may similarly allow the limit as $T \rightarrow \infty$ to obtain a measure $\mathbb{P}_{\lambda, \delta, q}^\tau = \lim_{T \rightarrow \infty} \mathbb{P}_{\lambda, \delta, q, T}^\tau$.

Let $G = \mathbb{Z}^d$. Restricting ourselves for convenience to the case of free boundary conditions, we define the percolation probability by

$$\theta(\lambda, \delta, q) = \mathbb{P}_{\lambda, \delta, q}(|C_0| = \infty),$$

and the critical point by

$$\lambda_c(\mathbb{Z}^d, q) = \sup\{\lambda : \theta(\lambda, 1, q) = 0\}.$$

In the special case $d = 1$, the random-cluster model has a property of self-duality that leads to the following conjecture.

Conjecture 4.3. *The continuum random-cluster model on $\mathbb{Z} \times \mathbb{R}$ with $q \geq 1$ has critical value $\lambda_c(\mathbb{Z}, q) = q$.*

It may be proved by standard means that $\lambda_c(\mathbb{Z}, q) \geq q$. See [25], Section 6.2, for the corresponding result on the discrete lattice \mathbb{Z}^2 .

The *continuum Potts model* on $G \times \mathbb{R}$ is given as follows. Let $q \in \{2, 3, \dots\}$. To each cluster of the random-cluster model with cluster-weighting factor q is assigned a ‘spin’ from the space $\Sigma = \{1, 2, \dots, q\}$, different clusters receiving independent spins. The outcome is a function $\sigma : V \times \mathbb{R} \rightarrow \Sigma$, and this is the spin-vector of a ‘continuum q -state Potts model’ with parameters λ and δ . When $q = 2$, we refer to the model as a continuum Ising model.

It may be seen that the law of the above spin model on $\Lambda = W \times [0, T]$ is given by

$$d\mathbb{P}(\sigma) = \frac{1}{Z} e^{\lambda L(\sigma)} d\mathbb{P}_{\Lambda, \delta}(D_\sigma),$$

where D_σ is the set of $(x, s) \in W \times [0, T]$ such that $\sigma(x, s-) \neq \sigma(x, s+)$, $\mathbb{P}_{\Lambda, \delta}$ is the law of a family of independent Poisson processes on the time-lines

$\{x\} \times [0, T]$, $x \in W$, with intensity δ , and

$$L(\sigma) = \sum_{\langle x, y \rangle \in E_W} \int_0^T 1_{\{\sigma(x, u) = \sigma(y, u)\}} du$$

is the aggregate Lebesgue measure of those subsets of adjacent time-lines on which the spins are equal. As usual, Z is an appropriate constant.

The continuum Ising model has arisen in the study by Aizenman, Klein, and Newman, [2], of the quantum Ising model with transverse field, as described in the next section.

5 The quantum Ising model

Aizenman, Klein, and Newman reported in [2] a representation of the quantum Ising model in terms of the $q = 2$ continuum random-cluster and Ising models. This was motivated in part by [15] and by earlier work referred to therein. We summarise this here, and we indicate how it may be used to study the property of entanglement in the quantum Ising model on \mathbb{Z} .

The quantum Ising model on a finite graph $G = (V, E)$ is given as follows. To each vertex $x \in V$ is associated a quantum spin- $\frac{1}{2}$ with local Hilbert space \mathbb{C}^2 . The Hilbert space \mathcal{H} for the system is therefore the tensor product $\mathcal{H} = \bigotimes_{x \in V} \mathbb{C}^2$. As basis for the copy of \mathbb{C}^2 labelled by x , we take the two eigenstates, denoted as $|+\rangle_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle_x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, of the Pauli operator

$$\sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

at the site x , with corresponding eigenvalues ± 1 . The other two Pauli operators with respect to this basis are the matrices

$$\sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (5.1)$$

In the following, $|\phi\rangle$ denotes a vector and $\langle\phi|$ its adjoint.

Let D be the set of $2^{|V|}$ basis vectors $|\eta\rangle$ for \mathcal{H} of the form $|\eta\rangle = \bigotimes_x |\pm\rangle_x$. There is a natural one-one correspondence between D and the space $\Sigma = \prod_{x \in V} \{-1, +1\}$. We shall sometimes speak of members of Σ as basis vectors, and of \mathcal{H} as the Hilbert space generated by Σ .

The Hamiltonian of the quantum Ising model with transverse field is the operator

$$H = -\frac{1}{2}\lambda \sum_{e=\langle x, y \rangle \in E} \sigma_x^{(3)} \sigma_y^{(3)} - \delta \sum_{x \in V} \sigma_x^{(1)}, \quad (5.2)$$

generating the operator $e^{-\beta H}$ where β denotes inverse temperature. Here, $\lambda, \delta \geq 0$ are the spin-coupling and external-field intensities, respectively. The Hamiltonian has a unique pure ground state $|\psi_G\rangle$ defined at zero-temperature (that is, in the limit as $\beta \rightarrow \infty$) as the eigenvector corresponding to the lowest eigenvalue of H .

Let

$$\rho_G(\beta) = \frac{1}{Z_G(\beta)} e^{-\beta H}, \quad (5.3)$$

where

$$Z_G(\beta) = \text{tr}(e^{-\beta H}) = \sum_{\eta \in \Sigma} \langle \eta | e^{-\beta H} | \eta \rangle.$$

It turns out that the matrix elements of $\rho_G(\beta)$ may be expressed as a type of ‘path integral’ with respect to the continuum random-cluster model on $G \times [0, \beta]$ with parameters λ, δ and $q = 2$. Let $\Lambda = V \times [0, \beta]$, write Ω_Λ for the configuration space of the latter model, and let $\phi_{G,\beta}$ be the appropriate continuum random-cluster measure on Ω_Λ (with free boundary conditions). For $\omega \in \Omega_\Lambda$, let S_ω denote the space of all functions $s : V \times [0, \beta] \rightarrow \{-1, +1\}$ that are constant on the clusters of ω , and let S be the union of the S_ω over $\omega \in \Omega_\Lambda$. Given ω , we may pick an element of S_ω uniformly at random, and we denote this random element as σ . We shall abuse notation by using $\phi_{G,\beta}$ to denote the ensuing probability measure on the coupled space $\Omega_\Lambda \times S$. For $s \in S$ and $W \subseteq V$, we write $s_{W,0}$ (respectively, $s_{W,\beta}$) for the vector $(s(x, 0) : x \in W)$ (respectively, $(s(x, \beta) : x \in W)$). We abbreviate $s_{V,0}$ and $s_{V,\beta}$ to s_0 and s_β , respectively.

The following representation of the matrix elements of $\rho_G(\beta)$ is obtained by expanding the exponential in (5.3), and it permits the use of random-cluster methods to study the matrix $\rho_G(\beta)$. For example, as pointed out in [2], it implies the existence of the low-temperature limits

$$\langle \eta' | \rho_G | \eta \rangle = \lim_{\beta \rightarrow \infty} \langle \eta' | \rho_G(\beta) | \eta \rangle, \quad \eta, \eta' \in \Sigma.$$

Theorem 5.4. [2] *The elements of the density matrix $\rho_G(\beta)$ are given by*

$$\langle \eta' | \rho_G(\beta) | \eta \rangle = \frac{\phi_{G,\beta}(\sigma_0 = \eta, \sigma_\beta = \eta')}{\phi_{G,\beta}(\sigma_0 = \sigma_\beta)}, \quad \eta, \eta' \in \Sigma. \quad (5.5)$$

This representation may be used to study entanglement in the quantum Ising model on G . Let $W \subseteq V$, and consider the *reduced density matrix*

$$\rho_G^W(\beta) = \text{tr}_{V \setminus W}(\rho_G(\beta)), \quad (5.6)$$

where the trace is performed over the Hilbert space $\mathcal{H}_{V \setminus W} = \bigotimes_{x \in V \setminus W} \mathbb{C}^2$ of the spins belonging to $V \setminus W$. By an analysis parallel to that leading to Theorem 5.4, we obtain the following.

Theorem 5.7. [27] *The elements of the reduced density matrix $\rho_G^W(\beta)$ are given by*

$$\langle \eta' | \rho_G^W(\beta) | \eta \rangle = \frac{\phi_{G,\beta}(\sigma_{W,0} = \eta, \sigma_{W,\beta} = \eta' \mid E)}{\phi_{G,\beta}(\sigma_0 = \sigma_\beta \mid E)}, \quad \eta, \eta' \in \Sigma_W, \quad (5.8)$$

where E is the event that $\sigma_{V \setminus W,0} = \sigma_{V \setminus W,\beta}$.

Let D_W be the set of $2^{|W|}$ vectors $|\eta\rangle$ of the form $|\eta\rangle = \bigotimes_{x \in W} |\pm\rangle_x$, and write \mathcal{H}_W for the space generated by D_W . Just as before, there is a natural one-one correspondence between D_W and the space $\Sigma_W = \prod_{x \in W} \{-1, +1\}$, and we shall regard \mathcal{H}_W as the Hilbert space generated by Σ_W .

We may write

$$\rho_G = \lim_{\beta \rightarrow \infty} \rho_G(\beta) = |\psi_G\rangle\langle\psi_G|$$

for the density matrix corresponding to the ground state of the system, and similarly

$$\rho_G^W = \text{tr}_{V \setminus W}(|\psi_G\rangle\langle\psi_G|) = \lim_{\beta \rightarrow \infty} \rho_G^W(\beta). \quad (5.9)$$

There has been extensive study of entanglement in the physics literature, see the references in [27]. The entanglement of the spins in W may be defined as follows.

Definition 5.10. *The entanglement of the vertex-set W relative to its complement $V \setminus W$ is the entropy*

$$S_G^W = -\text{tr}(\rho_G^W \log_2 \rho_G^W). \quad (5.11)$$

The behaviour of S_G^W , for general G and W , is not understood at present. Instead, we specialise here to the case of a finite subset of the one-dimensional lattice \mathbb{Z} . Let $m, L \geq 0$ and take $V = [-m, m+L]$ and $W = [0, L]$, viewed as subsets of \mathbb{Z} . We obtain G from V by adding edges between each pair $x, y \in V$ with $|x - y| = 1$. We write $\rho_m(\beta)$ for $\rho_G(\beta)$, and S_m^L for S_G^W . A key step in the study of S_m^L for large m is a bound on the norm of the difference $\rho_m^L - \rho_n^L$. For an operator A on \mathcal{H} , let

$$\|A\| = \sum_{\|\psi\|=1} |\langle \psi | A | \psi \rangle|,$$

where the supremum is over all $\psi \in \mathcal{H}_L$ with L^2 -norm 1.

Theorem 5.12. [27] *Let $\lambda, \delta \in (0, \infty)$ and write $\theta = \lambda/\delta$. There exist constants C, α, γ depending on θ and satisfying $\gamma > 0$ when $\theta < 1$ such that:*

$$\|\rho_m^L - \rho_n^L\| \leq \min\{2, CL^\alpha e^{-\gamma m}\}, \quad 2 \leq m \leq n < \infty. \quad (5.13)$$

One would expect that γ may be taken in such a manner that $\gamma > 0$ under the weaker assumption $\lambda/\delta < 2$, but this has not yet been proved (cf. Conjecture 4.3).

Inequality (5.13) is proved in [27] by the following route. Consider the random-cluster model with $q = 2$ on the space-time graph $\Lambda = V \times [0, \beta]$ with ‘partial periodic top/bottom boundary conditions’; that is, for each $x \in V \setminus W$, we identify the two vertices $(x, 0)$ and (x, β) . Let $\phi_{m,\beta}^p$ denote the associated random-cluster measure on Ω_Λ . To each cluster of $\omega \in \Omega_\Lambda$ we assign a random spin from $\{-1, +1\}$ in the usual manner, and we abuse notation by using $\phi_{m,\beta}^p$ for the measure governing both the random-cluster configuration and the spin configuration. Let $a_{m,\beta} = \phi_{m,\beta}^p(\sigma_{W,0} = \sigma_{W,\beta})$, noting that $a_{m,\beta} = \phi_{m,\beta}(\sigma_0 = \sigma_\beta \mid E)$ as in (5.8).

By Theorem 5.7,

$$\langle \psi | \rho_m^L(\beta) - \rho_n^L(\beta) | \psi \rangle = \frac{\phi_{m,\beta}^p(c(\sigma_{W,0})\overline{c(\sigma_{W,\beta})})}{a_{m,\beta}} - \frac{\phi_{n,\beta}^p(c(\sigma_{W,0})\overline{c(\sigma_{W,\beta})})}{a_{n,\beta}}, \quad (5.14)$$

where $c : \{-1, +1\}^W \rightarrow \mathbb{C}$ and

$$\psi = \sum_{\eta \in \Sigma_W} c(\eta) \eta \in \mathcal{H}_W.$$

The random-cluster property of ratio weak-mixing is used in the derivation of (5.13) from (5.14). At the final step of the proof of Theorem 5.12, the random-cluster model is compared with the continuum percolation model of Section 2, and the exponential decay of Theorem 5.12 follows by Theorem 2.1. A logarithmic bound on the entanglement entropy follows for sufficiently small λ/δ .

Theorem 5.15. [27] *Let $\lambda, \delta \in (0, \infty)$ and write $\theta = \lambda/\delta$. There exists $\theta_0 \in (0, \infty)$ such that: for $\theta < \theta_0$, there exists $K = K(\theta) < \infty$ such that*

$$S_m^L \leq K \log_2 L, \quad m \geq 0, \quad L \geq 2. \quad (5.16)$$

A stronger result is expected, namely that the entanglement S_m^L is bounded above, uniformly in L , whenever θ is sufficiently small, and perhaps for all $\theta < \theta_c$ where $\theta_c = 2$ is the critical point. See Conjecture 4.3 and the references

in [27]. There is no rigorous picture known of the behaviour of S_m^L for large θ , or of the corresponding quantity in dimensions $d \geq 2$, although Theorem 5.12 has a counterpart in this setting. Theorem 5.15 may be extended to the disordered system in which the intensities λ, δ are independent random variables indexed by the vertices and edges of the underlying graph, subject to certain conditions on these variables (cf. Theorem 2.4 and the preceding discussion).

6 The mean-field continuum model

The term ‘mean-field’ is often interpreted in percolation theory as percolation on either a tree (see [24], Chapter 10) or a complete graph. The latter case is known as the Erdős–Rényi random graph $G_{n,p}$, and this is the random graph obtained from the complete graph K_n on n vertices by deleting each edge with probability $1 - p$. The theory of $G_{n,p}$ is well developed and rather refined, see [10, 31], and particular attention has been paid to the emergence of the giant cluster for $p = \lambda/n$ and $\lambda \simeq 1$. A similar theory has been developed for the random-cluster model on K_n with parameters p, q , see [11, 25, 37].

Unless boundary conditions are introduced in the manner of [26, 28], the continuum random-cluster model on a tree may be solved exactly by standard means. We therefore concentrate here on the case of the complete graph K_n on n vertices. Let $\beta > 0$, and attach to each vertex the line $[0, \beta]$ with its endpoints identified; thus, the line forms a circle. We now consider the continuum random-cluster model on $K_n \times [0, \beta]$ with parameters $p = \lambda/n$, $\delta = 1$, and q . [The convention of setting $\delta = 1$ differs from that of [29] but is consistent with that adopted in earlier work on related models.]

Suppose that $q \geq 1$, so that we may use methods based on stochastic comparisons. It is natural to ask for the critical value $\lambda_c = \lambda_c(\beta, q)$ of λ above which the model possesses a giant cluster. This has been answered by Ioffe and Levit, [29], in the special case $q = 1$. Let $F(\beta, \lambda)$ be given by

$$F(\beta, \lambda) = \lambda[2(1 - e^{-\beta}) - \beta e^{-\beta}],$$

and let $\lambda_c = \lambda_c(\beta)$ be chosen so that $F(\beta, \lambda_c) = 1$.

Theorem 6.1. [29] *Let M be the maximal (one-dimensional) Lebesgue measure of the clusters of the process with parameters $\beta, p = \lambda/n, \delta = 1, q = 1$. Then, as $n \rightarrow \infty$,*

$$\frac{1}{n}M \rightarrow \begin{cases} 0 & \text{if } \lambda < \lambda_c, \\ \beta\pi & \text{if } \lambda > \lambda_c, \end{cases}$$

where $\pi = \pi(\beta, \lambda) \in (0, 1)$ when $\lambda > \lambda_c$, and the convergence is in probability.

When $\lambda > \lambda_c$, the density of the giant cluster is π , in that there is probability π that any given point of $K_n \times [0, \beta]$ lies in this giant cluster. The proof of Theorem 6.1 is simple to motivate. Let 0 be a vertex of K_n , and let I be the maximal cut-free interval of $0 \times [0, \beta]$ (viewed as a circle) containing the point 0×0 . Given I , the mean number of bridges leaving I is $\lambda|I|(n-1)/n \sim \lambda|I|$, where $|I|$ is the Lebesgue measure of I . One may thus approximate to the cluster at 0×0 by a branching process with mean family-size $\lambda E|I|$. It is elementary that $\lambda E|I| = F(\beta, \lambda)$, which is to say that the branching process is subcritical (respectively, supercritical) if $\lambda < \lambda_c$ (respectively, $\lambda > \lambda_c$). The details of the proof may be found in [29], and a further proof has appeared in [30]. The quantity π is of course the survival probability of the above branching process, and this may be calculated in the standard way on noting that $|I|$ is distributed as $\min\{U + V, \beta\}$ where U, V are independent, exponentially distributed, random variables with mean 1.

What is the analogue of Theorem 6.1 when $q \neq 1$? Indications are presented in [29] of the critical value when $q = 2$, and the problem is posed there of proving this value by calculations of the random-cluster type to be found in [11]. There is a simple argument that yields upper and lower bounds for the critical value for any $q \in [1, \infty)$. We present this next, and also explain our reason for believing the upper bound to be exact when $q \in [1, 2]$.

Consider the continuum random-cluster model on $K_n \times [0, \beta]$ with parameters $p = \lambda/n$, $\delta = 1$, and $q \in (0, \infty)$. Let

$$F_q(\beta, \lambda) = \frac{\lambda}{q^2} \cdot \frac{2e^{\beta q} - 2 + \beta q(q-2)}{e^{\beta q} + q - 1}, \quad (6.2)$$

noting that $F_1 = F$.

Theorem 6.3. *Let M_q be the maximal (one-dimensional) Lebesgue measure of the clusters of the process with parameters β , $p = \lambda/n$, $\delta = 1$, $q \in [1, \infty)$.*

- (i) *We have that $\lim_{n \rightarrow \infty} n^{-1}M_q = 0$ if $F_q < q^{-1}$, where the convergence is in probability.*
- (ii) *There exists $\pi_q = \pi_q(\beta, \lambda)$, satisfying $\pi_q > 0$ whenever $F_q > 1$, such that*

$$\liminf_{n \rightarrow \infty} P\left(\frac{1}{n}M_q \geq \beta\pi_q\right) \rightarrow 1.$$

The bound π_q may be calculated by a branching-process argument, in the same manner as was $\pi = \pi_1$, above. We conjecture that $n^{-1}M_q \rightarrow 0$ in probability if $F_q < 1$ and $q \in [1, 2]$. This conjecture is motivated by the evidence of [11] that, in the second-order phase transition occurring when

$q \in [1, 2]$, the location of the critical point is given by the branching-process approximation described in the sketch proof below. This amounts to the claim that the critical value $\lambda_c(q)$ of the continuum random-cluster model with cluster-weighting factor q satisfies

$$\lambda_c(q) = q^2 \frac{e^{\beta q} + q - 1}{2e^{\beta q} - 2 + \beta q(q - 2)}, \quad q \in [1, 2]. \quad (6.4)$$

This is implied by Theorem 6.1 when $q = 1$, and by the claim of [29] when $q = 2$. Note the relatively simple formula when $q = 2$,

$$\lambda_c(2) = \frac{2}{\tanh \beta}, \quad (6.5)$$

which might be termed the critical point of the *quantum random graph*. Dmitry Ioffe has pointed out that the exact calculation (6.5) may be derived from the results of [19, 21]. Results similar to those of Theorem 6.3 may be obtained for $q < 1$ also.

Sketch proof of Theorem 6.3. We begin with part (ii). The idea is to bound the process below by a random graph to which the results of [12, 30] may be applied directly. The bounding process is obtained as follows. First, we place the cuts on each of the time-lines $x \times [0, \beta]$, and we place no bridges. Thus, the cuts on a given time-line are placed in the manner of the continuum random-cluster model on that line. It may be seen that the number D of cuts on any given time-line has mass function

$$P(D = k) = \frac{e^{-\beta}}{Z} \cdot \frac{q^{k \vee 1} \beta^k}{k!}, \quad k \geq 0,$$

where $a \vee b = \max\{a, b\}$, and Z is the requisite constant,

$$Z = (q - 1)e^{-\beta} + e^{\beta(q-1)}.$$

It is an easy calculation that the maximal cut-free interval I containing the point 0×0 satisfies $E|I| = qF_q/\lambda$.

We next place edges between pairs of time-lines according to independent Poisson processes with intensity λ/q . We term the ensuing graph a ‘product random-cluster model’, and we claim that this model is dominated (stochastically) by the continuum random-cluster model. This may be seen in either of two ways: one may apply suitable comparison inequalities (see [25], Section 3.4) to a discrete approximation of $K_n \times [0, \beta]$ and then pass to the continuum limit, or one may establish it directly for the continuum model. Related material has appeared in [22, 41].

If this ‘product’ random-cluster model possesses a giant cluster, then so does the original random-cluster model. The former model may be studied either via the general techniques of [12, 30] for inhomogeneous random graphs, or using the usual branching process approximation. We follow the latter route here. In the limit as $n \rightarrow \infty$, the mean number of offspring of 0×0 approaches $(\lambda/q)E|I| = F_q$, so that the branching process is supercritical if $F_q > 1$. The claim of part (ii) follows.

For part (i) one proceeds similarly, but with λ/q replaced by λ and the domination reversed. \square

7 Acknowledgements

The author thanks Carol Bezuidenhout for encouraging him to persevere with continuum percolation and the contact model many years ago, and to Tobias Osborne and Petra Scudo for explaining the relationship between the quantum Ising model and the continuum random-cluster model. He is grateful to Svante Janson for their discussions of random-cluster models on complete graphs. Dima Ioffe has kindly pointed out the link between the quantum random graph and the work of [19, 21].

References

- [1] M. Aizenman and P. Jung. On the critical behavior at the lower phase transition of the contact process. 2006. arxiv:math.PR/0603227.
- [2] M. Aizenman, A. Klein, and C. M. Newman. Percolation methods for disordered quantum Ising models. In R. Kotecký, editor, *Phase Transitions: Mathematics, Physics, Biology, ...*, pages 129–137. World Scientific, Singapore, 1992.
- [3] K. Alexander. On weak mixing in lattice models. *Probability Theory and Related Fields*, 110:441–471, 1998.
- [4] K. Alexander. Mixing properties and exponential decay for lattice systems in finite volumes. *Annals of Probability*, 32:441–487, 2004.
- [5] E. Andjel. Survival of multidimensional contact process in random environments. *Boletim da Sociedade Brasileira de Matemática*, 23:109–119, 1992.

- [6] I. Benjamini and O. Schramm. Conformal invariance of Voronoi percolation. *Communications in Mathematical Physics*, 197:75–107, 1998.
- [7] C. E. Bezuidenhout and G. R. Grimmett. The critical contact process dies out. *Annals of Probability*, 18:1462–1482, 1990.
- [8] C. E. Bezuidenhout and G. R. Grimmett. Exponential decay for subcritical contact and percolation processes. *Annals of Probability*, 19:984–1009, 1991.
- [9] J. Bjornberg. 2007. In preparation.
- [10] B. Bollobás. *Random Graphs*. Cambridge University Press, 2nd edition, 2001.
- [11] B. Bollobás, G. R. Grimmett, and S. Janson. The random-cluster model on the complete graph. *Probability Theory and Related Fields*, 104:283–317, 1996.
- [12] B. Bollobás, S. Janson, and O. Riordan. The phase transition in inhomogeneous random graphs. *Random Structures and Algorithms*, 2007. arxiv:math.PR/0504589.
- [13] B. Bollobás and O. Riordan. The critical probability for random Voronoi percolation in the plane is $\frac{1}{2}$. *Probability Theory and Related Fields*, 136:417–468, 2006.
- [14] B. Bollobás and O. Riordan. *Percolation*. Cambridge University Press, 2006.
- [15] M. Campanino, A. Klein, and J. F. Perez. Localization in the ground state of the Ising model with a random transverse field. *Communications in Mathematical Physics*, 135:499–515, 1991.
- [16] R. Cerf. *The Wulff Crystal in Ising and Percolation Models*. Springer, Berlin, 2006.
- [17] L.-C. Chen and A. Sakai. Critical behavior and the limit distribution for long-range oriented percolation, I. 2007.
- [18] L.-C. Chen and N.-R. Shieh. Critical behavior for an oriented percolation with long-range interactions in dimension $d > 2$. *Taiwanese Journal of Mathematics*, 10:1345–1378, 2006.

- [19] T. Dorlas. Probabilistic derivation of a noncommutative version of Varadhan's theorem. 2002.
- [20] S. N. Ethier and T. G. Kurtz. *Markov Processes*. Wiley, New York, 1986.
- [21] M. Fannes, H. Spohn, and A. Verbeure. Equilibrium states for mean field models. *Journal of Mathematical Physics*, 21:355–358, 1980.
- [22] H.-O. Georgii and T. K  neth. Stochastic comparison of point random fields. *Journal of Applied Probability*, 34:868–881, 1997.
- [23] P. Grassberger and A. de la Torre. Reggeon field theory (Sch  gl's first model) on a lattice: Monte Carlo calculations of critical behaviour. *Annals of Physics*, 122:373–396, 1979.
- [24] G. R. Grimmett. *Percolation*. Springer, Berlin, 1999.
- [25] G. R. Grimmett. *The Random-Cluster Model*. Springer, Berlin, 2006.
- [26] G. R. Grimmett and S. Janson. Branching processes, and random-cluster measures on trees. *Journal of the European Mathematical Society*, 7:253–281, 2005.
- [27] G. R. Grimmett, T. J. Osborne, and P. F. Scudo. Entanglement in the quantum Ising model. 2007. arxiv:0704.2981.
- [28] O. H  ggstrom. The random-cluster model on a homogeneous tree. *Probability Theory and Related Fields*, 104:231–253, 1996.
- [29] D. Ioffe and A. Levit. Long range order and giant components of quantum random graphs. 2006.
- [30] S. Janson. On a random graph related to quantum theory. *Combinatorics, Probability and Computing*, 2007.
- [31] S. Janson, T. Łuczak, and A. Ruci  nski. *Random Graphs*. John Wiley, New York, 2000.
- [32] A. Klein. Extinction of contact and percolation processes in a random environment. *Annals of Probability*, 22:1227–1251, 1994.
- [33] A. Klein. Multiscale analysis in disordered systems: percolation and contact process in random environment. In G. R. Grimmett, editor, *Disorder in Physical Systems*, pages 139–152. Kluwer, Dordrecht, 1994.

- [34] G. Lawler. *Conformally Invariant Processes in the Plane*. American Mathematical Society, 2005.
- [35] T. M. Liggett. *Interacting Particle Systems*. Springer, Berlin, 1985.
- [36] T. M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer, Berlin, 1999.
- [37] M. Łuczak and T. Łuczak. The phase transition in the cluster-scaled model of a random graph. *Random Structures and Algorithms*, 28:215–246, 2006.
- [38] R. Meester and R. Roy. *Continuum Percolation*. Cambridge University Press, 1996.
- [39] C. M. Newman and S. Volchan. Persistent survival of one-dimensional contact processes in random environments. *Annals of Probability*, 24:411–421, 1996.
- [40] M. D. Penrose. *Random Geometric Graphs*. Oxford University Press, 2003.
- [41] C. J. Preston. Spatial birth-and-death processes. *Bulletin of the International Statistical Institute*, 46:371–391, 405–408, 1975.
- [42] A. Sakai and G. Slade. Critical points for spread-out self-avoiding walk, percolation and the contact process above the upper critical dimensions. *Probability Theory and Related Fields*, 132:438–470, 2005.
- [43] O. Schramm. Conformally invariant scaling limits. In *Proceedings of the International Congress of Mathematicians, Madrid, 2006*. European Mathematical Society, Zurich, 2007.
- [44] G. Slade. *The Lace Expansion and its Applications*. Springer, Berlin, 2006.
- [45] S. Smirnov. Towards conformal invariance of 2D lattice models. In *Proceedings of the International Congress of Mathematicians, Madrid, 2006*. European Mathematical Society, Zurich, 2007.
- [46] W. Werner. Random planar curves and Schramm–Loewner evolutions. In J. Picard, editor, *Ecole d’Eté de Probabilités de Saint Flour XXXII–2002*, pages 107–195. Springer, Berlin, 2004.